Received: *September 10, 1998*, Accepted: *September 14, 1998*

$AdS_5 \times S^5$ black hole metric at $\mathcal{O}(\alpha'^3)$

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Abstract: We compute $\mathcal{O}(\alpha'^3)$ corrections to the $AdS_5 \times S^5$ black hole metric. We find that the radius of the S^5 depends on the radial AdS_5 coordinate. This completes the computation of Gubser, Klebanov and Tseytlin (hep-th/9805156). The fact that the metric no longer factorizes should modify the value of the Wilson line at finite temperature and the glueball mass spectrum.

Keywords: Superstring Vacua, Black Holes in String Theory, Supergravity Models.

*Supported by the Alexander-von-Humboldt Foundation.

Sparked by Maldacena's conjecture [1] there has recently been a resurgence of interest in supergravity in anti-de-Sitter space. In the simplest case the type IIB vacuum is the direct product $AdS_5 \otimes S^5$ [2]. Many of the subsequent papers on various aspects of Maldacena's conjecture were based on the leading order supergravity actions. However, since the conjecture refers to the complete string theory, one should consider the string corrections to the 10D supergravity action. The first corrections occur at order $(\alpha')^3$ and have been known for a long time [3]. Taking theses effects into account, as shown by Banks and Green [4], does not change the metric in the extremal $AdS_5 \otimes S^5$ case. This was subsequently verified to all orders in α' in [5]. In the non-extremal case this is however no longer true and one is faced with the task to compute the corrections to the metric and the other background fields, such as the dilaton and the anti-symmetric tensor field. This problem was addressed recently by Gubser, Klebanov and Tseytlin [6]. Their analysis, which was restricted to the AdS_5 part of the metric, turns out to be sufficient for the computation of the corrections of the free energy. However, the corrections to the full ten-dimensional metric has not been found in [6], as has been erroneously assumed in several subsequent papers. Specifically, the dynamics of the conformal factor was not found. Here we reconsider the issue for the full ten-dimensional metric and show that at $\mathcal{O}(\alpha'^3)$ it no longer factorizes.

The starting point for the analysis is the low-energy supergravity action in the Einstein frame

$$S = \frac{N^2}{16\pi^7}\int d^{10}x\sqrt{-g}\Big\{\mathcal{R} - \frac{1}{2}(\partial\phi)^2 + \gamma e^{-\frac{3}{2}\phi}W - \frac{1}{4\cdot 5!}\frac{1}{N^2}F_5^2\Big\}\,, \tag{1}$$

where we have defined

$$\gamma \equiv \frac{1}{8}\zeta(3)(g_s N)^{-3/2}\,. \tag{2}$$

In the Maldacena limit $(g_s N)^{1/2} \sim \alpha'$. Note that the normalization is such that $F_5 \sim N$. For details, in particular for a discussion on the form $W \sim \mathcal{C}^4$ ($\mathcal{C}$ is the Weyl tensor) of the eighth derivative term and the subtleties with the self-duality of F_5 we refer to [6] and references quoted therein.

We make the most general ansatz compatible with the symmetries of the problem:

$$ds^2 = H^2(r)\Big(K^2(r)d\tau^2 + P^2(r)dr^2 + \sum_{i=1}^{3} dx_i^2\Big) + L^2(r)d\Omega_5^2\,. \tag{3}$$

We first show that it is not possible to keep the radius of the S^5 fixed to 1 (if H is fixed), i.e. $L(r) = 1$ is not a solution to the equations of motion. We will then find the correct solution of the equations of motion following from the ansatz (3). As the authors of [4] and [6] we assume that the vielbein components of F_5 do not change.

After rescaling $ds^2 \to \Lambda^2 ds^2$ the part of the action containing Λ is

$$\begin{aligned} S \supset &\int d^{10}x\sqrt{g}\Lambda^{10}\left\{\Lambda^{-2}R - 18\,\Lambda^{-3}\nabla^2\Lambda - 54\,\Lambda^{-4}(\nabla\Lambda)^2 + \gamma\Lambda^{-8}W\right\} \\ = &\int d^{10}x\sqrt{g}\left\{\Lambda^8\mathcal{R} + 72\,\Lambda^6(\nabla\Lambda)^2 + \gamma\Lambda^2 W\right\}\,. \end{aligned} \tag{4}$$

Here $\mathcal{R}$ is the curvature scalar of the metric $ds^2 = ds_1^2 \oplus d\Omega_5^2$, i.e. the metric (3) with $L(r) = 1$. We have neglected terms $\mathcal{O}(\gamma^2)$. They will not enter the argument. Due to the direct sum structure we have $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 = \mathcal{R}_1 + 20$, the latter part coming from S^5 ($\mathcal{R}_{S^n} = n(n-1)$).

Consider now the equation of motion for Λ,

$$8\,\Lambda^7(\mathcal{R}_1 + 20) - 6 \cdot 72\,\Lambda^5(\nabla\Lambda)^2 - 2 \cdot 72\,\Lambda^6\nabla^2\Lambda + 2\,\gamma\Lambda W = 0\,. \tag{5}$$

The solution for Λ will be of the form

$$\Lambda = \Lambda^{(0)} + \gamma\Lambda^{(1)} + \dots\,, \tag{6}$$

where $\Lambda^{(0)}$ and $\Lambda^{(1)}$ are both $\mathcal{O}(\alpha'^0)$. We likewise expand

$$\mathcal{R}_1 = \mathcal{R}_1^{(0)} + \gamma\mathcal{R}_1^{(1)} + \dots \tag{7}$$

and

$$W = W^{(0)} + \gamma W^{(1)} + \dots \tag{8}$$

If $\Lambda \equiv 1$ is a solution, the equation of motion for Λ will be satisfied if

$$\mathcal{R}_1^{(0)} + 20 = 0 \qquad \text{and} \qquad 4\mathcal{R}_1^{(1)} + W^{(0)} = 0 \tag{9}$$

hold. In fact $\mathcal{R}_1^{(0)} = -20$ but the latter equation is not satisfied, because

$$W^{(0)} = 180\frac{r_0^{16}}{r^{16}} \qquad \text{and} \qquad \mathcal{R}_1^{(1)} = 180\frac{r_0^{16}}{r^{16}} \tag{10}$$

for the metric given in [6]. This completes the proof that the ten-dimensional metric is not of the form $ds_1^2 \oplus d\Omega_5^2$.

Next we will solve the equations of motion following from the general ansatz (3) which shows explicitly that a direct product geometry is not a solution. For ease of comparison with [6] we choose the following parameterization for the functions $H(r), K(r), P(r), L(r)$:

$$\begin{aligned} H(r) &= r\,, & K(r) &= e^{a(r)+4b(r)}\,,\\ P(r) &= e^{b(r)}\,, & L(r) &= e^{c(r)}\,. \end{aligned} \tag{11}$$

In terms of these functions the lowest, i.e. zeroth order in α', contribution to the action is $(' = \partial_r)$

$$\begin{aligned} S = \int dr \Big\{ & 4r^5\left(5 - 2e^{-8c}\right)e^{a+5b+3c} \\ & + \left(-2r(2 + ra') + 10r^3c'(a' + 4b' + 2c')\right)e^{a+3b+5c} \\ & -2\left(r^3(a' + 4b' + 5c')e^{a+3b+5c}\right)'\Big\}\,. \end{aligned} \tag{12}$$

We have dropped an overall factor $\frac{N^2}{\pi^7}\mathrm{Vol}(S^5)\mathrm{Vol}(\mathbf{R}^{3,1}) = \frac{N^2}{\pi^4}\mathrm{Vol}(\mathbf{R}^{3,1})$. The expression for the W term is too long to be reproduced here.

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Since we can consistently find solutions to $\mathcal{O}(\gamma)$ only, we write

$$a(r) = a^{(0)}(r) + \gamma a^{(1)}(r) \tag{13}$$

and likewise for $b(r)$ and $c(r)$, suppressing higher order terms in γ. Perturbation in γ requires the zeroth order solutions. They are

$$\begin{aligned} a^{(0)}(r) &= -\log(r^2) + \frac{5}{2}\log(r^4 - r_0^4) \\ b^{(0)}(r) &= -\frac{1}{2}\log(r^4 - r_0^4) \\ c^{(0)}(r) &= 0\,. \end{aligned} \tag{14}$$

The equations of motion for the first order (in γ) corrections get contributions from the term $\propto \gamma W$ in the action eq.(1). They are, up to a factor γ,

$$540(19r_0^4 - 16r^4)\frac{r_0^{12}}{r^{13}}\,, \qquad 540(79r_0^4 - 64r^4)\frac{r_0^{12}}{r^{13}}\,, \qquad 900\frac{r_0^{16}}{r^{13}} \tag{15}$$

for the equation for $a(r)$, $b(r)$ and $c(r)$, respectively. The equations can now be easily solved with the ansatz

$$a^{(1)}(r) = a_0 + a_1\frac{r_0^4}{r^4} + a_2\frac{r_0^8}{r^8} + a_3\frac{r_0^{12}}{r^{12}} + \ldots \tag{16}$$

and likewise for $b^{(1)}(r)$ and $c^{(1)}(r)$. It turns out that higher powers in $\frac{r_0}{r}$ beyond the ones displayed will not contribute. The results are

$$\begin{aligned} a^{(1)}(r) &= -\frac{1625}{8}\frac{r_0^4}{r^4} - 175\frac{r_0^8}{r^8} + \frac{10005}{16}\frac{r_0^{12}}{r^{12}} \\ b^{(1)}(r) &= \frac{325}{8}\frac{r_0^4}{r^4} + \frac{1075}{32}\frac{r_0^8}{r^8} - \frac{4835}{32}\frac{r_0^{12}}{r^{12}} \\ c^{(1)}(r) &= \frac{15}{32}\frac{r_0^8}{r^8}\left(1 + \frac{r_0^4}{r^4}\right). \end{aligned} \tag{17}$$

a_0, which is undetermined and related to rescaling of time, has been set to zero [6]. The equation for the first correction of the dilaton $\phi = -\log(g_s) + \gamma\phi^{(1)} + \ldots$ is the same as in [6] and leads to

$$\phi^{(1)}(r) = -\frac{45}{8}\Big(\frac{r_0^4}{r^4} + \frac{1}{2}\frac{r_0^8}{r^8} + \frac{1}{3}\frac{r_0^{12}}{r^{12}}\Big)\,. \tag{18}$$

We can also give the necessary reparameterization that transforms the five-dimensional AdS_5 part of the metric as computed here to the one computed in [6]:

$$r \to r\Big[1 - \gamma\frac{25}{32}\Big(\frac{r_0^8}{r^8} + \frac{r_0^{12}}{r^{12}}\Big)\Big]\,, \qquad r_0 \to r_0\Big(1 - \frac{25}{16}\gamma\Big)\,. \tag{19}$$

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The resulting metric is

$$ds^2 = e^{-10/3\nu(r)}H^2(r)\Big(K^2(r)d\tau^2 + P^2(r)dr^2 + \sum_{i=1}^{3} dx_i^2\Big) + e^{2\nu(r)}d\Omega_5^2\,, \tag{20}$$

where $H(r) = r$, $K(r)$ and $P(r)$ are as in [6] and to $\mathcal{O}(\gamma)$

$$\nu(r) = \gamma\frac{15}{32}\frac{r_0^8}{r^8}(1+\frac{r_0^4}{r^4}) \tag{21}$$

There are several applications of our result. First of all, we have reconsidered the corrections of thermodynamic quantities, following ref.[6]. It turns out that the correction to the free energy does not change. For comparison we give some results. For the temperature we find

$$2\pi T = 2r_0\Big(1+\frac{265}{16}\gamma\Big)\,. \tag{22}$$

The action is

$$I = \frac{N^2}{4\pi^2}\beta\mathrm{Vol}(\mathbf{R}^3)(r_{\max}^4 - r_0^4)\left(1 - \frac{325}{4}\gamma\left[\frac{r_0^4}{r_{\max}^4} + \mathcal{O}(\frac{r_0^8}{r_{\max}^8})\right]\right)\,. \tag{23}$$

For the free energy we find

$$F = -\frac{\pi^2}{8}N^2V_3T^4(1+15\gamma)\,, \tag{24}$$

which agrees with the result in [6]. For an independent argument why the value of the free energy does not change, see the note added in [6].

One can also see that the scalar glueball spectrum (without KK modes) [7] is unchanged. The reason is that inclusion of the L^2 factor in (3) does not influence the relevant equations of motions. However there will be corrections to the other glue-ball masses and to KK glueballs. Likewise, the coefficients of the $\mathcal{O}(\gamma)$ corrections to the Wilson loop at finite temperature [8] will change.

Acknowledgments

We want to thank Arkady Tseytlin for useful correspondence. S.T. also thanks D. Maison for a clarifying discussion. Our work is supported by GIF, the German-Israeli Foundation for Scientific Research and by the European Commission TMR programme ERBFMRX-CT96-0045.

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